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Solution for 'Topics in complex analysis'

(05/11/2025)

H 8.1 (Practice with Hadamard products)
Find the Hadamard products for:

a) $e^z - 1$. b) $\cos(\pi z)$.

Solution H 8.1:
a) The function $f(z) = e^z - 1$ has simple zeros at $z = 2\pi i n$ for $n \in \mathbb{Z}$, and no other zeros. We have $|f(z)| \le e^{|z|} + 1 \le 2e^{|z|}$, so f has an order of growth ≤ 1 and taking large $z \in \mathbb{R}_{>0}$ we see that indeed the order of growth is $\rho_f = 1$. Thus the Hadamard factorization theorem shows that there are constants $a, b \in \mathbb{C}$ such that

$$e^z - 1 = e^{az+b} z \prod_{n \in \mathbb{Z}\setminus\{0\}} E_1\Big(\frac{z}{2\pi i n}\Big) = e^{az+b} z \prod_{n \in \mathbb{Z}\setminus\{0\}} \Big(1 - \frac{z}{2\pi i n}\Big) e^{\frac{z}{2\pi i n}} = e^{az+b} z \prod_{n=1}^{\infty} \Big(1 + \frac{z^2}{4\pi^2 n^2}\Big),$$

where we paired the factors for n and $-n$ (recall that local normal convergence allows us to freely rearrange the factors). From the series expansion we have $\frac{e^z-1}{z} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!}$, and from the display above this is $e^{az+b} Q(z)$, where $Q(z)$ denotes the product of Weierstrass factors, which is an entire function with $Q(0) = 1$ and is even. Thus $Q(z) = 1 + O(z^2)$, while $e^{az+b} = e^b(1 + az) + O(z^2)$ near $z = 0$. Matching the first two terms of the corresponding series expansions gives $e^b(1 + az) = 1 + \frac{z}{2}$, so $e^b = 1$ and $a = \frac{1}{2}$. We conclude that

$$e^z - 1 = e^{z/2} z \prod_{n=1}^{\infty} \Big(1 + \frac{z^2}{4\pi^2 n^2}\Big).$$

b) The function $f(z) = \cos(\pi z)$ has simple zeros at $z = \frac{1}{2} + n$ for $n \in \mathbb{Z}$, and no other zeros. From $|\cos(\pi z)| = \big|\frac{e^{i\pi z} + e^{-i\pi z}}{2}\big| \le e^{\pi|z|}$ we see that f has an order of growth ≤ 1, and taking $z = iw$ for large $w \in \mathbb{R}$ shows that in fact the order of growth is $\rho_f = 1$. Thus as before Hadamard's theorem gives

$$\cos(\pi z) = e^{az+b} \prod_{n \in \mathbb{Z}} E_1\Big(\frac{2z}{2n+1}\Big) = e^{az+b} \prod_{n \in \mathbb{Z}} \Big(1 - \frac{2z}{2n+1}\Big) e^{\frac{2z}{2n+1}} = e^{az+b} \prod_{n=0}^{\infty} \Big(1 - \frac{4z^2}{(2n+1)^2}\Big),$$

where we paired n with $-1-n$. Since $\cos(\pi z)$ and the product of Weierstrass factors are both even functions, we see that $a = 0$, and plugging in $z = 0$ gives $e^b = 1$, hence

$$\cos(\pi z) = \prod_{n=0}^{\infty} \Big(1 - \frac{4z^2}{(2n+1)^2}\Big).$$

Remark: Observe that this matches Exercise H 6.4 b).

H 8.2 (Picard's little theorem for functions of finite order)
Show that if f is an entire function with finite order of growth that omits two values, then f is constant.

Remark: We will soon prove this result for any entire function (Picard's little theorem).
Hint: If f misses a, then $f(z) - a$ is of the form $e^{P(z)}$ where P is a polynomial.

Solution H 8.2:
Suppose that f misses two distinct values $a, b \in \mathbb{C}$. Since $f(z) - a$ has finite order of growth and does not vanish, Hadamard's theorem shows that $f(z) - a = e^{P(z)}$ for some polynomial P. Observe that the image of e^z is $\mathbb{C} \setminus \{0\}$, since for $z = re^{i\theta}$ with $r > 0$ we have $z = e^{\log r + i\theta}$.
If P is constant, then so is f and we are done. Otherwise it has degree at least 1, so the image of $P(z)$ is all of $\mathbb{C}$ by the fundamental theorem of algebra. Since $a \neq b$, we conclude that there exists $w \in \mathbb{C}$ such that $e^{P(w)} = b - a$, hence $f(w) = e^{P(w)} + a = b$, which is a contradiction. Thus f is constant.

H 8.3 (A transcendental equation)
Show that the equation $e^z = z$ has infinitely many solutions $z \in \mathbb{C}$.

Solution H 8.3:
The function $f(z) = e^z - z$ has order of growth $\rho_f = 1$. Assume by contradiction that it has only finitely many zeros $a_1, a_2, \dots, a_N$. Then applying Hadamard's theorem we obtain

$$e^z - z = e^{cz+d} \prod_{n=1}^{N} \Big(1 - \frac{z}{a_n}\Big) e^{\frac{z}{a_n}} = e^{z\left(c + \sum_{n=1}^{N} \frac{1}{a_n}\right)} e^d \prod_{n=1}^{N} \Big(1 - \frac{z}{a_n}\Big)$$

for some constants $c, d \in \mathbb{C}$. Thus there is a polynomial $Q(z)$ and a constant $a \in \mathbb{C}$ such that $e^z - z = e^{az} Q(z)$, so

$$1 - ze^{-z} = e^{(a-1)z} Q(z) \qquad \text{for all } z \in \mathbb{C}.$$

Inserting absolute values and taking a limit as $z = x \in \mathbb{R}$ grows, we obtain $1 = \lim_{x\to\infty} e^{(\mathrm{Re}(a)-1)x} |Q(x)|$. This limit is 0 if $\mathrm{Re}\,(a) < 1$ and ∞ if $\mathrm{Re}\,(a) > 1$, so we have $a = 1 + it$ for $t \in \mathbb{R}$. Then the previous equation becomes $1 = \lim_{x\to\infty} |Q(x)|$, which implies $Q \equiv 1$ is a constant polynomial. Thus

$$1 - ze^{-z} = e^{itz} \qquad \text{for all } z \in \mathbb{C}.$$

Now taking a limit with $z = itx$ as $x \in \mathbb{R}$ grows, we get $\lim_{x\to\infty}(1 - itxe^{-itx}) = \lim_{x\to\infty} e^{-t^2 x}$. If $t \neq 0$ then $\lim_{x\to\infty} e^{-t^2 x} = 0$, but $|itxe^{-itx}| = |tx| \to \infty$ as $x \to \infty$, which is a contradiction. Thus $t = 0$, and we get $-ze^{-z} = 0$ for all $z \in \mathbb{C}$, which is a contradiction. Therefore $e^z - z$ has infinitely many zeros.

H 8.4 (Non-integral order of growth)
Show that if f is entire with finite order of growth ρ_f which is non-integral, then f has infinitely many zeros.

Solution H 8.4:
Suppose $f \not\equiv 0$ has finitely many zeros. Then applying Hadamard's theorem, we conclude as in the previous solution that $f(z) = e^{R(z)} Q(z)$ for some polynomials R and Q, where R has degree $\leq k := \lfloor \rho_f \rfloor$. Indeed, R is the sum of the polynomial P (of degree $\leq k$) from the exponential factor in Hadamard's theorem and the terms $\sum_{n=1}^{N} \sum_{m=1}^{k} \frac{1}{m} \big(\frac{z}{a_n}\big)^m$ from the exponential part of the Weierstrass factors $E_k(z/a_n)$, where $a_1, a_2, \dots, a_N$ are the finitely many zeros of f. But any polynomial has order of growth 0, i.e. for any $\varepsilon > 0$ there are $A, B > 0$ such that

$$|Q(z)| \leq Ae^{B|z|^\varepsilon} \qquad \text{for all } z \in \mathbb{C}.$$

Denoting $\ell := \deg(R)$, observe $|R(z)| \le C|z|^\ell$ for all $z \in \mathbb{C}$. We conclude that

$$|f(z)| = |e^{R(z)}Q(z)| = e^{\mathrm{Re}(R(z))}|Q(z)| \le e^{|R(z)|}|Q(z)| \le De^{E|z|^{\ell+\varepsilon}} \qquad \text{for all } z \in \mathbb{C}.$$

Thus $\rho_f \le \ell + \varepsilon$ for any $\varepsilon > 0$, hence $\rho_f \le \ell$. But as a consequence of Hadamard's theorem we had $\ell := \deg(R) \le k := \lfloor \rho_f \rfloor$, so $\rho_f \le \lfloor \rho_f \rfloor$ and we must have $\rho_f \in \mathbb{Z}$. This finishes the proof.

H 8.5 (Non-vanishing derivatives)
Suppose f is entire and never vanishes, and that none of its higher derivatives ever vanish. Prove that if f also has finite order of growth, then $f(z) = e^{az+b}$ for some constants $a, b \in \mathbb{C}$.

Solution H 8.5:
By Hadamard's theorem, if f has finite order of growth and never vanishes then $f(z) = e^{P(z)}$ for some polynomial P. Then $f'(z) = P'(z)e^{P(z)}$ has the same zeros as the polynomial $P'(z)$. If P' is not a constant polynomial then the fundamental theorem of algebra shows that it has a root, hence $f'(z)$ has a zero, which contradicts the non-vanishing of derivatives of f. Thus $P'(z) = a \in \mathbb{C}$ is constant, so $P(z) = az + b$ must be a linear polynomial, as desired.

Remark: We only need the non-vanishing of f and f'.

H 8.6 (Order of growth in terms of Taylor coefficients)
Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be an entire function of finite order of growth. Then the order of growth of f is intimately linked with the growth of the coefficients a_n as $n \to \infty$.

a) Show that $g(z) = \sum_{n=0}^\infty \frac{z^n}{(n!)^\alpha}$ describes an entire function of order $1/\alpha$, for any $\alpha > 0$.

Remark: You may want to skip this item and return to it later, since it is a little complicated.

b) Suppose $|f(z)| \le Ae^{a|z|^\rho}$ for every $z \in \mathbb{C}$, where $\rho > 0$. Then

$$\limsup_{n\to\infty} |a_n|^{1/n} n^{1/\rho} < \infty. \tag{1}$$

Hint: Use Cauchy's inequality

$$|a_n| \le \sup_{|z|=r} \frac{|f(z)|}{r^n} \le \frac{A}{r^n} e^{ar^\rho}$$

and the fact that the function $u^{-n}e^{u^\rho}$, for $u > 0$, attains its minimum value $e^{n/\rho}(\rho/n)^{n/\rho}$ at $u = n^{1/\rho}/\rho^{1/\rho}$. Then choose r in terms of n to achieve this minimum.

c) Conversely, if (1) holds, then for any $\varepsilon > 0$ we have $|f(z)| \le A_\varepsilon e^{a_\varepsilon |z|^{\rho+\varepsilon}}$ for every $z \in \mathbb{C}$.

Hint: Note that for $|z| = r$,

$$|f(z) - a_0| \le \sum_{n=1}^\infty \frac{c^n r^n}{n^{n/\rho}} \le \sum_{n=1}^\infty \frac{c^n r^n}{(n!)^{1/\rho}}$$

for some constant c, since $n^n \ge n!$. This yields a reduction to item a).

Solution H 8.6:
a) Let us first show that $|g(z)| \le Ae^{a|z|^{1/\alpha+\varepsilon}}$. It suffices to show this under the assumption that $|z|$ is sufficiently large in terms of α (simply adjust the constant A to cover the smaller $|z|$). Observe that for $n \ge N \in \mathbb{N}$ we have $n! \ge N! \cdot N^{n-N}$, so

$$|g(z)| \le \sum_{n=0}^{N-1} \frac{|z|^n}{(n!)^\alpha} + \frac{|z|^N}{(N!)^\alpha} \sum_{n=N}^\infty \left(\frac{|z|}{N^\alpha}\right)^{n-N}.$$

If say $N^\alpha \geq 2|z|$ and $|z| \geq 2$ then we conclude that

$$|g(z)| \leq \sum_{n=0}^{N-1} |z|^n + \frac{|z|^N}{(N!)^\alpha} \cdot \frac{1}{1-\frac{|z|}{N^\alpha}} \leq |z|^N + 2\frac{|z|^N}{(N!)^\alpha} \leq 3|z|^N.$$

We choose $N = \lceil (2|z|)^{1/\alpha} \rceil$, so that $N < (2|z|)^{1/\alpha} + 1 \leq 2(2|z|)^{1/\alpha}$. Thus for all $|z| \geq 2$ we have

$$|g(z)| \leq 3|z|^N \leq 3|z|^{2(2|z|)^{1/\alpha}} = 3e^{c|z|^{1/\alpha} \log |z|} \leq A_\varepsilon e^{a_\varepsilon |z|^{1/\alpha+\varepsilon}}$$

for any $\varepsilon > 0$, so the order of growth of g satisfies $\rho_g \leq 1/\alpha$, as desired.
To show that $\rho_g = 1/\alpha$, we give a lower bound for $|g(x)|$ for large $x \in \mathbb{R}_{>0}$. Indeed, for any $N \in \mathbb{N}$ we have

$$g(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^\alpha} \geq \frac{x^N}{(N!)^\alpha} \geq \Big(\frac{x}{N^\alpha}\Big)^N,$$

by positivity of the terms and the inequality $N^N \geq N!$. Inspired by the previous argument, we choose the parameter $N = \lceil (x/3)^{1/\alpha} \rceil < (x/3)^{1/\alpha} + 1 \leq (x/2)^{1/\alpha}$ for $x \in \mathbb{R}_{>0}$ sufficiently large in terms of α, to conclude that

$$g(x) \geq \Big(\frac{x}{\lceil (x/3)^{1/\alpha} \rceil^\alpha}\Big)^{(x/3)^{1/\alpha}} \geq 2^{(x/3)^{1/\alpha}} \geq e^{cx^{1/\alpha}}.$$

Thus no inequality of the form $|g(z)| \leq Ae^{B|z|^{1/\alpha-\varepsilon}}$ may hold for every $z \in \mathbb{C}$ when $\varepsilon > 0$. Therefore the order of growth of g is $\rho_g \geq \frac{1}{\alpha}$, and this finishes the proof that $\rho_g = \frac{1}{\alpha}$.

b) By Cauchy's inequality (itself a simple consequence of Cauchy's formula) and the assumption $|f(z)| \leq Ae^{a|z|^\rho}$ for every $z \in \mathbb{C}$, for each $r > 0$ and $n \in \mathbb{N}$ we obtain

$$|a_n| \leq \sup_{|z|=r} \frac{|f(z)|}{r^n} \leq \frac{A}{r^n} e^{ar^\rho}.$$

Denoting $h(u) = u^{-n} e^{u^\rho}$ for $u > 0$, we have $h'(u) = \frac{-n}{u^{n+1}} e^{u^\rho} + \frac{\rho u^{\rho-1}}{u^n} e^{u^\rho}$, so the minimum value of $h(u)$ for $u > 0$ is attained when $n = \rho u^\rho$, hence $u = (n/\rho)^{1/\rho}$. Since we would like to minimize the bound above, let us choose $r = a^{-1/\rho}(n/\rho)^{1/\rho}$ to conclude that for all $n \in \mathbb{N}$ we have

$$|a_n| \leq Aa^{n/\rho}(a^{1/\rho}r)^{-n} e^{(a^{1/\rho}r)^\rho} = A\Big(\frac{ea\rho}{n}\Big)^{n/\rho}.$$

Therefore $|a_n|^{1/n} n^{1/\rho} \leq Aea\rho$ is uniformly bounded in terms of n, and we obtain the desired conclusion $\limsup_{n\to\infty} |a_n|^{1/n} n^{1/\rho} < \infty$.

c) If $\limsup_{n\to\infty} |a_n|^{1/n} n^{1/\rho} < \infty$ then there is $c > 0$ such that $|a_n| \leq \frac{c^n}{n^{n/\rho}}$ for all $n \in \mathbb{N}$. Thus if $|z| = r > 0$ then

$$|f(z)| \leq |a_0| + \sum_{n=1}^{\infty} \frac{c^n r^n}{n^{n/\rho}} \leq |a_0| + \sum_{n=1}^{\infty} \frac{c^n r^n}{(n!)^{1/\rho}},$$

since $n^n \geq n!$. Using item a) with $\alpha = 1/\rho$, the right hand side is $|a_0| - 1 + g(r)$. Since g has order of growth ρ, for any $\varepsilon > 0$ there are constants $A_\varepsilon, a_\varepsilon > 0$ such that

$$|f(z)| \leq A_\varepsilon e^{a_\varepsilon r^{\rho+\varepsilon}} = A_\varepsilon e^{a_\varepsilon |z|^{\rho+\varepsilon}},$$

where we adjusted A_ε to incorporate the constant $|a_0| - 1$. This finishes the proof.

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